Indefinite Nilsolitons and Einstein Solvmanifolds

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Abstract

Homogeneous Einstein manifolds are among the most studied objects in geometry, and in particular the only known non-compact examples with definite metrics are solvable manifolds. Thanks to the work of many authors (see in particular [2]), we know that the construction of solvable Lie groups admitting an Einstein left-invariant Riemannian metric is related to the construction of Ricci soliton left-invariant metrics on nilpotent Lie groups (called nilsolitons). In this talk, we will describe the relationship between nilsolitons and Einstein left-invariant metrics on solvable Lie groups in the context of pseudo-Riemannian geometry.

More precisely, a nilsoliton is a nilpotent Lie algebra $\mathfrak{g}$ with a metric such that $\mathrm{Ric} = \lambda \,\mathrm{id} + D$, with D a derivation. For indefinite metrics, this determines four different geometries, according to whether λ and D are zero or not. We will explain with examples the greater flexibility of the indefinite case compared to the Riemannian setting.

It was shown in [3] that every pseudo-Riemannian nilsoliton with $\lambda \neq 0$ and $\operatorname{tr} D \neq 0$ determines an Einstein solvable Lie algebra. We will extend this result to a more general setting.

In particular, for each of the four geometries, we will show that under suitable assumptions it is possible to extend the nilsoliton metric to an Einstein solvmanifold of the form $\mathfrak{g} \rtimes \mathbb{R}^k$. Conversely, we will introduce a large class of indefinite Einstein solvmanifolds of the form $\mathfrak{g} \rtimes \mathbb{R}^k$, called *pseudo-Iwasawa*, that determine a nilsoliton metric on $\mathfrak{g}$ by restriction. It turns out that the metrics constructed in [3] are pseudo-Iwasawa in our sense. We will illustrate with examples that, unlike in the Riemannian case, one cannot establish a correspondence between the full classes of Einstein solvmanifolds and nilsolitons.

This is a joint work with Diego Conti (Università Milano-Bicocca), and the results are described in [1].

References

[1] D. Conti, F. A. Rossi: *Indefinite nilsolitons and Einstein solvmanifolds.* Preprint arXiv:2105.09209.

[2] J. Lauret: *Ricci soliton homogeneous nilmanifolds.* Math. Ann., **319** n.4, 715–733, (2001).

[3] Z. Yan: *Pseudo-Riemannian Einstein metrics on noncompact homogeneous spaces.* J. Geom., **111**, (2020).