International Fall Workshop in Geometry and Physics

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August 21, 2021

Short course: *Some topics on deep learning and numerical analysis*

Lecture 1: *Deep learning as optimal control and structure preserving deep learning.*

Deep learning neural networks have recently been interpreted as discretisations of an optimal control problem subject to an ordinary differential equation constraint. A large amount of progress made in deep learning has been based on heuristic explorations, but there is a growing effort to mathematically understand the structure in existing deep learning methods and to design new approaches preserving (geometric) structure in neural networks. The (discrete) optimal control point of view to neural networks offers an interpretation of deep learning from a numerical analysis perspective and opens the way to mathematical insight.

We review the first order conditions for optimality, and the conditions ensuring optimality after discretisation. We explain the connection to partitioned Runge-Kutta methods for Hamiltonian systems. The differential equation setting lends itself to learning additional parameters such as the time discretisation. We compare these deep learning algorithms numerically in terms of induced flow and generalisation ability.

We furter discuss a number of interesting directions of current and future research in structure preserving deep learning. Some deep neural networks can be designed to have desirable properties such as invertibility and group equivariance or can be adapted to problems of manifold value data. If time permits, we discuss neural networks that are designed so to guarantee stability and contractivity and that can be used to solve inverse problems in imaging. This lecture is built on material taken from [3, 4, 5].

Lecture 2: *An introduction to shape analysis and deep learning for optimal reparametrizations of shapes.*

Shape analysis is a mathematical approach to problems of pattern and object recognition and has developed considerably in the last decade. The use of shapes is natural in applications where it is interesting to compare curves or surfaces independently of their parametrisation. Considering a smooth setting where the parametrized curves or surfaces belong to an infinite dimensional Riemannian manifold, one defines the corresponding shapes to be equivalence classes of curves differing only by their parametrization. Under appropriate assumptions, the Riemannian metric can be used to obtain a meaningful measure of distance on the space of shapes.

One computationally efficient approach to shape analysis is based on the Square Root Velocity Transform, and we have proposed a generalisation of this approach to shapes on Lie groups and homogeneous manifolds.

A computationally demanding task for approximating shape distances is finding the optimal reparametrization. The problem can be phrased as an optimisation problem on the infinite dimensional group of orientation preserving diffeomorphisms $\mathrm{Diff}^+(\Omega)$, where Ω is the domain where the curves or surfaces are defined. In the case of curves, one robust approach to compute optimal reparametrizations is based on dynamic programming, [11], but this method seems difficult to generalize to surfaces.

We consider here a method inspired by a "Riemannian" gradient descent approach obtained by representing the gradient $\operatorname{grad} E$ in terms of an othonormal basis of $T_{\mathrm{id}}\mathrm{Diff}^+(\Omega)$ and projecting $\operatorname{grad} E$ on a finite dimensional subspace. The approximations are obtained composing in succession a number of elementary diffeomorphism equal to the number of iterations, and optimizing on few parameters at the time. This method can be improved by optimising simultaneously over a larger number of parameters in an approach reminisent of deep learning. This algorithm is motivated by results in [1], about the controllability of the group of diffeomorphisms, via the composition of a finite number of elementary diffeomorphisms.

This lecture is built on material taken from [6, 7, 8, 12].

References

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