The supersymmetric nonlinear sigma model as a geometric variational problem

XXIX International Fall Workshop in Geometry and Physics

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September 9, 2021

universität wien

FWF Der Wissenschaftsfonds.

Outline

Harmonic maps

- Let $(M, h_{\alpha\beta})$ and (N, g_{ij}) be closed Riemannian manifolds.
- Let $\phi\colon M \to N$ be a smooth map.
- We define its energy (in physics: Polyakov action) by

$$E(\phi) = \int_M |d\phi|^2 dM = \int_M \frac{\partial \phi^i}{\partial x^\alpha}\frac{\partial \phi^j}{\partial x^\beta} g_{ij}(\phi) h^{\alpha\beta} dM\,.$$

- $E(\phi)$ is invariant under conformal transformations on M if $\dim M = 2$.
- **Harmonic maps** are critical points of $E(\phi)$ which satisfy

$$\tau(\phi) = \nabla_{e_\alpha} d\phi(e_\alpha) = 0, \qquad \tau(\phi) \in \Gamma(\phi^* TN).$$

  In terms of local coordinates we have

$$\Delta_M \phi^i + \Gamma^i_{jk} h_{\alpha\beta} \frac{\partial \phi^j}{\partial x_\alpha}\frac{\partial \phi^k}{\partial x_\beta} = 0.$$

- Harmonic maps are a semi-linear second order **elliptic** PDE.

Existence of harmonic maps

- Use the L^2-**gradient flow**:

$$\frac{\partial \phi_t}{\partial t} = \tau(\phi_t), \qquad \phi(\cdot, 0) = \phi_0 \,. \tag{1}$$

Theorem (Eells - Sampson, 1964)

Let M and N be closed Riemannian manifolds and assume that the ***sectional curvature of*** N ***is non-positive****.*
Then (1) has a unique smooth solution $\phi_t \in C^\infty(M \times [0,\infty), N)$ for arbitrary $\phi_0 \in C^\infty(M, N)$. For $t \to \infty$, it converges to a smooth harmonic map in $C^\infty(M, N)$.

- What happens if we weaken the condition $K_N \leq 0$?
- Eells - Wood have shown: There does not exist a harmonic map $\phi\colon \mathbb{T}^2 \to \mathbb{S}^2$ with $\deg \phi \pm 1$ regardless of the metrics on M and N.

Wave maps

- Let $(M, h_{\alpha\beta})$ be a globally hyperbolic Lorentzian manifold and (N, g_{ij}) be a complete Riemannian manifold.
- Let $\phi\colon M \to N$ be a map.
- We define the energy functional

$$E(\phi) = \frac{1}{2}\int_M |d\phi|^2 dM.$$

- $E(\phi)$ is invariant under conformal transformations on M if $\dim M = 2$.
- **Wave maps** are critical points of $E(\phi)$, which satisfy

$$\mathrm{Tr}_g \nabla d\phi = 0, \qquad \mathrm{Tr}_g \nabla d\phi \in \Gamma(\phi^* TN).$$

- In terms of local coordinates we have

$$\Box_M \phi^i + \Gamma^i_{jk} h_{\alpha\beta} \frac{\partial \phi^j}{\partial x_\alpha} \frac{\partial \phi^k}{\partial x_\beta} = 0.$$

- Wave maps are a semi-linear second order **hyperbolic** PDE.

Existence of wave maps

- Suppose $M = \mathbb{R}^{1,1}$. For smooth initial data: Use method of characteristics (Gu, 1980) and obtain a global smooth solution!
- Result was extended by Shatah (1988) to distributional initial data.

Theorem (Shatah - Struwe, 2002)

Let $M = \mathbb{R}^{1,n}$ and $(\phi, \phi_t)\big|_{t=0} = (\phi_0, \phi_1) \in H^{\frac{n}{2}} \times H^{\frac{n}{2}-1}(\mathbb{R}^n, TN)$. Suppose N is complete, without boundary and has bounded curvature. Assume $n \geq 4$. Then there exists a constant $\epsilon > 0$ such that for any $(\phi_0, \phi_1) \in H^{\frac{n}{2}} \times H^{\frac{n}{2}-1}(\mathbb{R}^n, TN)$ satisfying

$$\|\phi_0\|_{H^{\frac{n}{2}}} + \|\phi_1\|_{H^{\frac{n}{2}-1}} < \epsilon$$

there exists a unique global solution $\phi \in C^0(\mathbb{R}, H^{\frac{n}{2}}) \cap C^1(\mathbb{R}, H^{\frac{n}{2}-1})$ of the wave map system that preserves any higher regularity of the initial data.

Spin geometry on Riemannian manifolds

Let (M, h) be a closed Riemannian spin manifold.

- Spinors are sections in the spinor bundle: $\psi \in \Gamma(\Sigma M)$
- The construction of spinors involves both the **Riemannian metric** h, in addition we have to fix a **spin structure**.
- The Dirac operator $\not{\partial} : \Gamma(\Sigma M) \to \Gamma(\Sigma M)$ is given by $\not{\partial} := h^{\alpha\beta} e_\alpha \cdot \nabla^{\Sigma M}_{e_\beta}$.
  Here, $\{e_\alpha\}$ denotes a local basis of TM and $\cdot$ Clifford multiplication.
- Clifford relations $X \cdot Y + Y \cdot X = -2h(X, Y)$ for all $X, Y \in TM$.
- The Dirac operator on a Riemannian manifold is
  - linear and of first order
  - elliptic
  - self-adjoint in $L^2(\Sigma M)$.
- Schroedinger-Lichnerowicz formula: $\not{\partial}^2 = \nabla^*\nabla + \frac{Scal}{4}$.
- A spinor is called **harmonic** if $\not{\partial}\psi = 0$.
- There do not exist harmonic spinors on $\mathbb{S}^2$ regardless of the metric.

Spin geometry on Lorentzian manifolds

- Now, let (M, h) be a globally hyperbolic spin manifold.
- The Dirac operator is defined as $\not{\partial} := h^{\alpha\beta} e_\alpha \cdot \nabla^{\Sigma M}_{e_\beta}$.
- Note that $i\not{\partial}$ is self-adjoint in L^2, that is

$$\int_M \langle i\not{\partial}\psi, \eta\rangle_{\Sigma M}\, dM = \int_M \langle \psi, i\not{\partial}\eta\rangle_{\Sigma M}\, dM$$

  for all $\psi, \eta \in \Gamma(\Sigma M)$.
- The Dirac operator on a Lorentzian manifold is
  - linear and of first order
  - hyperbolic
  - anti self-adjoint in $L^2(\Sigma M)$.
- There exists a Green's function for $\not{\partial}\psi = 0$, no restrictions from the geometry of M.

The action functional of the SUSY σ-model in coordinates

The action functional of the supersymmetric nonlinear sigma model as you find it in the physics literature:

$$S_\sigma^P(\phi,\psi)=\frac{1}{2}\int_M\Big(h^{\alpha\beta}g_{ij}\frac{\partial\phi^i}{\partial x_\alpha}\frac{\partial\phi^j}{\partial x_\beta}+\epsilon^{\alpha\beta}B_{ij}\frac{\partial\phi^i}{\partial x_\alpha}\frac{\partial\phi^j}{\partial x_\beta}+g_{ij}\langle\psi^i,(\not{D}\psi)^j\rangle$$
$$+A_{ijk}\frac{\partial\phi^k}{\partial x_\alpha}\langle e_\alpha\cdot\psi^i,\psi^j\rangle-\frac{1}{6}R_{ijkl}\langle\psi^i,\psi^k\rangle\langle\psi^j,\psi^l\rangle$$
$$-\frac{1}{3}\nabla_i A_{jkl}\langle\psi^i,\psi^k\rangle\langle\psi^j,\psi^l\rangle-\frac{1}{3}A_{imj}A^m_{kl}\langle\psi^i,\psi^k\rangle\langle\psi^j,\psi^l\rangle\Big)dM.$$

- $\phi\colon(M,h)\to(N,g)$ is a map and $\dim M=2$.
- $\psi^i\in\Gamma(\Sigma M), i=1,\dots,\dim N$ are spinors with "Dirac operator" $\not{D}$
- R_{ijkl} curvature tensor on N
- B_{ij} two-form on N
- A_{ijk} "torsion" three-form on N

The action functional for geometers

Invariant formulation of the action functional

$$S_\sigma(\phi,\psi)=\frac{1}{2}\int_M(|d\phi|^2+2\phi^*B+\langle\psi,\not{D}^{Tor}\psi\rangle-\frac{1}{6}\langle R^N_{Tor}(\psi,\psi)\psi,\psi\rangle)dM.$$

- The differential of ϕ can be thought of $d\phi\in\Gamma(T^*M\otimes\phi^*TN)$.
- ψ is a vector spinor, $\psi\in\Gamma(\Sigma M\otimes\phi^*TN)$.
- B is a two-form on N.
- R^N_{Tor} it the curvature tensor on N for a metric connection with torsion.
- $\not{D}^{Tor}$ is the twisted Dirac operator on $\Sigma M\otimes\phi^*TN$ for a metric connection with torsion on N.
- Difference to physics: Our spinors are not anticommuting!
- Challenging mathematical problem: Under which conditions do critical points of $S_\sigma(\phi,\psi)$ exist?

Some remarks on the action functional

- In physics the precise form of the action functional is fixed by symmetries (SUSY, diffeomorphisms on M).
- $S_\sigma(\phi,\psi)$ is invariant under conformal transformations on M in the case of a two-dimensional domain.
- For the moment, we assume that (M,h) is Riemannian.
- Due to the Dirac-Term: $S_\sigma(\phi,\psi)$ is unbounded.
- Analytic point of view:

$$S_\sigma(\phi,\psi) \leq C \int_M (|d\phi|^2 + |\psi|^4 + |\nabla^{\Sigma M}\psi|^{\frac{4}{3}}) dM$$

- The mathematical analysis of $S_\sigma(\phi,\psi)$ requires to apply tools from spin geometry and geometric analysis.

Dirac-harmonic maps

- $\left.\begin{array}{l}\text{Harmonic maps}\\ \text{Harmonic spinors}\end{array}\right\} \Rightarrow ?$ (Physics: Supersymmetry)
- **Dirac-harmonic maps** (introduced by Chen, Jost et. al in 2004) are critical points of

$$S_{DH}(\phi,\psi) = \frac{1}{2}\int_M (|d\phi|^2 + \langle\psi, \not{D}\psi\rangle)dM,$$

  where $\psi \in \Gamma(\Sigma M \otimes \phi^* TN)$ is a **vector spinor** and $\not{D}$ the **twisted Dirac operator**.
- The connection on $\Sigma M \otimes \phi^* TN$ is denoted by $\tilde{\nabla}$, thus $\not{D} := e_\alpha \cdot \tilde{\nabla}_{e_\alpha}$ with $\{e_\alpha\}$ a local basis of TM and $\cdot$ denotes Clifford multiplication.
- The critical points of $S_{DH}(\phi,\psi)$ are given by

$$\tau(\phi) = \frac{1}{2}R^N(\psi, e_\alpha \cdot \psi)d\phi(e_\alpha) := \mathcal{R}(\phi,\psi), \qquad \not{D}\psi = 0.$$

Regularity of Dirac-harmonic maps on surfaces I

- We apply Nash's theorem to embed N into some $\mathbb{R}^q$ isometrically.
- We obtain a system $(\phi\colon M \to \mathbb{R}^q, \psi\colon M \to \Sigma M \otimes \mathbb{R}^q)$

$$
\begin{aligned}
-\Delta\phi &= C(d\phi, d\phi) + D(d\phi, \psi, \psi), \\
\partial\!\!\!/\psi &= E(\psi, d\phi).
\end{aligned}
$$

- The quantities C, D, E only depend on geometric data.
- For $\dim M = 2$ the function space for weak Dirac-harmonic maps is $\chi(M,N) := \{(\phi,\psi) \in W^{1,2}(M,N) \times W^{1,\frac{4}{3}}(M, \Sigma M \otimes \phi^* TN)\}$.

Theorem (Chen - Jost - Wang - Li, 2004)

Let $(\phi,\psi) \in \chi(M,N)$ be a weak Dirac-harmonic map and $\dim M = 2$. *If ϕ is continuous, then (ϕ,ψ) is smooth.*

- How do we obtain the continuity of ϕ?

Regularity of Dirac-harmonic maps on surfaces II

Theorem (Rivière, 2007)

Let D be the unit disc in $\mathbb{R}^2$ and fix $q \in \mathbb{N}$.
For every $A = A^i{}_j, 1 \leq i, j \leq q$ in $L^2(D, \mathfrak{so}(q) \otimes \mathbb{R}^2)$(that is $A^i{}_j = -A^j{}_i$), a weak solution $\phi \in W^{1,2}(D, \mathbb{R}^q)$ of

$$-\Delta\phi = A \cdot \nabla\phi$$

is continuous.

- Generalization of the classic Wente-Lemma.
- It is crucial that A is antisymmetric!
- By application of Rivière's theorem we obtain the continuity of ϕ.

Removable singularity theorem for Dirac-harmonic maps

Theorem (Removable singularity theorem, Chen et. al, 2004)

For $U \subset M$ let (ϕ, ψ) be a Dirac-harmonic map, which is C^∞ on $U \setminus \{p\}$ for a $p \in U$. If

$$\int_U \left(|d\phi|^2 + |\psi|^4\right) d\mu \leq C$$

then (ϕ, ψ) extends to a C^∞ solution on the whole of U.

- Proof uses local energy estimates.
- Proof uses scaling: If $(\phi(x), \psi(x))$ is a Dirac-harmonic map, then also

$$\tilde{\phi} := \phi(rx), \qquad \tilde{\psi} := \sqrt{r}\psi(rx)$$

for some $r > 0$.

Metric connections with torsion I

- We want to study target manifolds having a metric connection with torsion.
- For every affine connection there exists a $(2,1)$-Tensor A such that

$$\nabla_X Y = \nabla^{LC}_X Y + A(X,Y)$$

for all vector fields $X, Y \in \Gamma(TN)$.

- We demand that the connection ∇^{Tor} is still metric, that is for all vector fields X, Y, Z we have

$$\partial_X \langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle.$$

- Thus, the endomorphisms $A(X,\cdot)$ has to be "skew-adjoint"

$$\langle A(X,Y), Z\rangle = -\langle Y, A(X,Z)\rangle.$$

Metric connections with torsion II

- We obtain a $(3,0)$ tensor by setting $A_{XYZ} = \langle A(X,Y),Z\rangle$.
- The curvature tensors satisfy

$$\begin{aligned} R^{Tor}(X,Y)Z =& R^{LC}(X,Y)Z + (\nabla^{LC}_X A)(Y,Z) - (\nabla^{LC}_Y A)(X,Z) \\ &+ A(X,A(Y,Z)) - A(Y,A(X,Z)). \end{aligned}$$

- The space of all possible torsion tensors is given by

$$\mathcal{T}(T_pN) = \left\{A \in \otimes^3 T_p^*N \mid A_{XYZ} = -A_{XZY} \;\; \forall X,Y,Z \in T_pN\right\}.$$

- We set: $c_{12}(A)(Z) = A_{\partial_{y^i}\partial_{y^i}Z}$ with ∂_{y^i} being a basis of TN.

Classification of orthogonal connections with torsion

Theorem (Cartan, 1924)

Let $\dim N \geq 3$*. The space* $\mathcal{T}(T_pN)$ *has the following irreducible decomposition:*

$$\mathcal{T}(T_pN) = \mathcal{T}_1(T_pN) \oplus \mathcal{T}_2(T_pN) \oplus \mathcal{T}_3(T_pN)$$

This decomposition is orthogonal and given by:

$$\mathcal{T}_1(T_pN) = \{A \in \mathcal{T}(T_pN) \mid \exists V \; A_{XYZ} = \langle X, Y\rangle\langle V, Z\rangle - \langle X, Z\rangle\langle V, Y\rangle\}$$
$$\mathcal{T}_2(T_pN) = \{A \in \mathcal{T}(T_pN) \mid A_{XYZ} = -A_{YXZ} \;\; \forall X, Y, Z\}$$
$$\mathcal{T}_3(T_pN) = \{A \in \mathcal{T}(T_pN) \mid A_{XYZ} + A_{YZX} + A_{ZXY} = 0 \textit{ and } c_{12}(A)(Z) = 0\}$$

For $\dim N = 2$*, we have* $\mathcal{T}(T_pN) = \mathcal{T}_1(T_pN)$*.*

- $\mathcal{T}_1(T_pN)$ is called "*Vectorial* torsion"
- $\mathcal{T}_2(T_pN)$ is called "*Totally antisymmetric* torsion"
- $\mathcal{T}_3(T_pN)$ is called "*Cartan type* torsion"

Dirac-harmonic maps with torsion (Branding, 2015)

- Assume that we have a metric connection with torsion on N.

$$
\begin{aligned}
S_{Tor}(\phi,\psi) =& \frac{1}{2}\int_M (|d\phi|^2 + \langle\psi, \not{D}^{Tor}\psi\rangle)dM \\
=& \frac{1}{2}\int_M (|d\phi|^2 + \langle\psi, \not{D}\psi\rangle + \langle\psi, A(d\phi(e_\alpha), e_\alpha\cdot\psi)\rangle)dM
\end{aligned}
$$

- The energy functional $S_{Tor}(\phi,\psi)$ is real valued.
- $\not{D}^{Tor}$ is still self-adjoint.
- The critical points of $S_{Tor}(\phi,\psi)$ (called *Dirac-harmonic maps with torsion*) are given by

$$
\tau(\phi) = \mathcal{R}(\phi,\psi) + F^{Tor}(\phi,\psi), \qquad \not{D}^{Tor}\psi = 0.
$$

- $F^{Tor}(\phi,\psi)$ has the same analytic structure as $\mathcal{R}(\phi,\psi)$.
- If dim $M = 2$ a weak solution $(\phi,\psi)\in\chi(M,N)$ is smooth and the removable singularity theorem also holds.

Dirac-harmonic maps with curvature term

- Now, we study the functional (without torsion)

$$S_c(\phi,\psi)=\frac{1}{2}\int_M(|d\phi|^2+\langle\psi,\not{D}\psi\rangle-\frac{1}{6}R_{ijkl}\langle\psi^i,\psi^k\rangle\langle\psi^j,\psi^l\rangle)dM.$$

- Its critical points are given by

$$\tau(\phi)=\mathcal{R}(\phi,\psi)-\frac{1}{12}\langle(\nabla R^N)^\sharp(\psi,\psi)\psi,\psi\rangle,$$
$$\not{D}\psi=\frac{1}{3}R^N(\psi,\psi)\psi.$$

- This is a coupled system of two non-linear equations!
- Interesting limit: For ϕ being trivial, the equation for ψ is known as *Spinorial Weierstrass representation* for CMC surfaces in $\mathbb{R}^3$, also appears as *Thirring model* in quantum field theory.

Regularity of Dirac-harmonic maps with curvature term

- Suppose $\dim M = 2$ and set

$$\chi(M,N) := \{(\phi,\psi) \in W^{1,2}(M,N) \times W^{1,\frac{4}{3}}(M, \Sigma M \otimes \phi^* TN)\}$$

- A weak solution $(\phi,\psi) \in \chi(M,N)$ is smooth (Branding, 2014).
  - Adam's inequality on Morrey spaces
  - Regularity theory of Topping-Sharp/Riviere:

Suppose that $\phi \in W^{1,2}(D_1, \mathbb{R}^q)$ is a weak solution of

$$-\Delta\phi = A \cdot \nabla\phi + f, \qquad f \in L^p(D_1, \mathbb{R}^q),$$

where $A \in L^2(D_1, \mathfrak{so}(q) \otimes \mathbb{R}^2)$ and $p \in (1,2)$. Then $\phi \in W^{2,p}_{loc}(D_1)$.

- The regularity result for Dirac-harmonic maps with curvature term was later generalized to all dimensions by Jost, Liu, Zhu.

Energy estimates for the full model

Suppose we have a system of the form (assuming $\dim M = 2$)

$$\begin{aligned} \tau(\phi) =& A(\phi)(d\phi, d\phi) + B(\phi)(d\phi, \psi, \psi) + C(\phi)(\psi, \psi, \psi, \psi), \\ \not{D}\psi =& E(\phi)(d\phi)\psi + F(\phi)(\psi, \psi)\psi, \end{aligned} \tag{2}$$

where the quantities A, B, C, E, F only depend on geometric data.

Theorem (ϵ-regularity theorem, Branding, 2015)

For a smooth solution of (2) *with small energy* $\int_D (|d\phi|^2 + |\psi|^4) d\mu < \epsilon$ *we have*

$$\begin{aligned} |d\phi|_{W^{1,p}(\tilde{D})} &\leq C(\tilde{D}, p)(|d\phi|_{L^2(D)} + |\psi|^2_{L^4(D)}), \\ |\nabla\psi|_{W^{1,p}(\tilde{D})} &\leq C(\tilde{D}, p)|\psi|_{L^4(D)} \end{aligned}$$

for all $\tilde{D} \subset D, p > 1$.

Removable singularity theorem

Theorem (Branding, 2015, (later also by Jost et. al))

For $U \subset M$ let (ϕ, ψ) be a Dirac-harmonic map with curvature term, which is C^∞ on $U \setminus \{p\}$ for a $p \in U$. If

$$\int_U \left(|d\phi|^2 + |\psi|^4\right) d\mu \leq C$$

then (ϕ, ψ) extends to a C^∞ solution on the whole of U.

- Based on ideas from Sacks-Uhlenbeck for harmonic maps.
- Applies local energy estimates.
- Proof uses scaling: If $(\phi(x), \psi(x))$ is a Dirac-harmonic map with curvature term, then also

$$\tilde{\phi} := \phi(r + rx), \qquad \tilde{\psi} := \sqrt{r}\psi(r + rx)$$

for some $r > 0$.

A Liouville Theorem for the domain being a closed surface

- The nodal set of solutions of $\not{D}\psi = \frac{1}{3}R^N(\psi,\psi)\psi$ is discrete on closed surfaces due to a result of Bär.
- If (ϕ,ψ) is a smooth Dirac-harmonic map with curvature term, then

$$C_N \int_M (|d\phi|^2 + |\psi|^4)dM \geq \pi\chi(M) + 2\pi N(\psi)$$

with the constant $C_N := \max\{\frac{|R^N|^2_{L^\infty}}{6}, |R^N|_{L^\infty}\}$ and

$$N(\psi) = \sum_{p\in M, |\psi|(p)=0} n_p,$$

where n_p denotes the order of vanishing.
Method: Calculate $\Delta \log|\psi|^2$ and integrate over M.
- Hence, if $\chi(M) > 0$ and if $\int_M (|d\phi|^2 + |\psi|^4)dM$ is "too small", then (ϕ,ψ) is trivial.
- This result can also be obtained by the Sobolev embedding theorem.

A Liouville Theorem for stationary solutions

- For $\dim M \geq 3$ weak Dirac-harmonic maps with curvature term live in

$$\chi(M,N) := W^{1,2}(M,N)\times W^{1,\frac{4}{3}}(M,\Sigma M\otimes\phi^* TN)\times L^4(M,\Sigma M\otimes\phi^* TN).$$

- A weak Dirac-harmonic map with curvature term is called *stationary* if it is also a critical point of $S_c(\phi,\psi)$ with respect to domain variations.

Theorem (Branding, 2016)

Let $M = \mathbb{R}^n, \mathbb{H}^n$ with $\dim M \geq 3$ and suppose that (ϕ,ψ) is a stationary Dirac-harmonic maps with curvature term satisfying

$$\int_{\mathbb{R}^n}(|d\phi|^2 + |\nabla^{\Sigma M}\psi|^{\frac{4}{3}} + |\psi|^4)dM < \infty.$$

If N has positive sectional curvature then ϕ is constant and ψ vanishes identically.

The energy-momentum tensor of $S_c(\phi,\psi)$

- The functional $S_c(\phi,\psi)$ is invariant under diffeomorphisms on M.
- Due to Noether's theorem we get a conserved quantity, which is the energy momentum tensor

$$\begin{aligned}S(X,Y) =& 2\langle d\phi(X), d\phi(Y)\rangle - h(X,Y)|d\phi|^2 \\ &+ \frac{1}{2}\langle\psi, X\cdot\nabla_Y^{\Sigma M\otimes\phi^*TN}\psi + Y\cdot\nabla_X^{\Sigma M\otimes\phi^*TN}\psi\rangle \\ &- \frac{1}{6}h(X,Y)\langle R^N(\psi,\psi)\psi,\psi\rangle.\end{aligned}$$

- If (ϕ,ψ) is a Dirac-harmonic map with curvature term, then $S(X,Y)$ is divergence-free.
- In the case of harmonic maps ($\psi = 0$) the energy-momentum tensor can be used to derive powerful monotonicity formulas.
- Due to the Dirac-term it is difficult to find "interesting" monotonicity formulas in the framework of Dirac-harmonic maps.

General Existence Results

Dirac-harmonic maps and their extensions have nice properties, but do they exist?

- The functional $S_\sigma(\phi, \psi)$ is unbounded both from above and from below.
  - Cannot use the *direct method of the calculus of variations*.
  - Cannot directly use the well-known existence scheme of *Sacks and Uhlenbeck* for harmonic maps in dimension two, which is

    $$E_\alpha(\phi) = \int_M (1 + |d\phi|^2)^\alpha dM$$

    for some $\alpha > 1$.

    Some progress by Jost et al. using this approach recently.
  - Cannot directly apply the *heat flow method*.
- The Dirac operator is of first order, there is no maximum principle.
- Currently only few (partial) existence results in the Riemannian setting are available.
- What about the case when M is a Lorentzian manifold?

Dirac-wave maps from two-dimensional Minkowski space

- Let M be two-dimensional Minkowski space with global coordinates (t,x) and (N,g) a Riemannian manifold.
- Technical difficulty: The geometric scalar product on ΣM is indefinite.
- The action functional for Dirac-wave maps is given by

$$S(\phi,\psi)=\frac{1}{2}\int_{\mathbb{R}^{1,1}}(|d\phi|^2+\langle\psi,i\not{D}\psi\rangle)dM.$$

- The critical points are given by

$$\begin{aligned}\frac{\nabla}{\partial t}d\phi(\partial_t)-\frac{\nabla}{\partial x}d\phi(\partial_x)=&\frac{1}{2}R^N(\psi,i\partial_t\cdot\psi)d\phi(\partial_t)\\&-\frac{1}{2}R^N(\psi,i\partial_x\cdot\psi)d\phi(\partial_x),\\ \partial_t\cdot\tilde{\nabla}_{\partial_t}\psi=&\partial_x\cdot\tilde{\nabla}_{\partial_x}\psi.\end{aligned}$$

- For smooth initial data: Existence result due to Han.

An explicit solution on two-dimensional Minkowski space

- A spinor $\psi \in \Gamma(\Sigma M)$ is called *twistor spinor* if it satisfies

$$P_X\psi := \nabla^{\Sigma M}_X \psi + \frac{1}{n} X \cdot \partial\!\!\!/ \psi = 0.$$

- In two-dimensional Minkowski space twistor spinors are of the form

$$\psi(x) = \psi_1 + x \cdot \psi_2,$$

where ψ_1, ψ_2 are constant spinors.

Let $\phi\colon \mathbb{R}^{1,1} \to N$ be a wave map. We set

$$\psi := e_\alpha \cdot \chi \otimes d\phi(e_\alpha),$$

where χ is a twistor spinor. Then the pair (ϕ, ψ) is a Dirac-wave map, that is uncoupled:

$$\tau(\phi) = 0 = \frac{1}{2} R^N(\psi, ie_\alpha \cdot \psi) d\phi(e_\alpha), \qquad D\!\!\!/ \psi = 0.$$

Conserved energies for Dirac-wave maps

- Let (ϕ,ψ) be a Dirac-wave map from $\mathbb{R}^{1,1}$.
- By $|\cdot|_\beta$ we denote the definite scalar product on $\Sigma\mathbb{R}^{1,1}$.
- We find $\Box|\psi|^2_\beta=0$, where $\Box:=\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}$. We obtain

$$|\psi|^2_\beta\leq C.$$

- Set $e(\phi):=\frac{1}{2}(|d\phi(\partial_t)|^2+|d\phi(\partial_x)|^2)$. Then we find

$$\Box(e(\phi)-\langle\tilde{\nabla}_{\partial_t}\psi,i\partial_t\cdot\psi\rangle)=0.$$

- Useful fact: If $f\colon\mathbb{R}^{1,1}\to\mathbb{R}$ solves $\Box f=0$ then

$$\frac{d}{dt}\int_{\mathbb{R}}(|\frac{\partial f}{\partial t}|^4+|\frac{\partial f}{\partial x}|^4+6|\frac{\partial f}{\partial t}|^2|\frac{\partial f}{\partial x}|^2)dx=0.$$

Existence of Dirac-wave maps from $\mathbb{R}^{1,1}$

Theorem (Branding, 2017)

Let $\mathbb{R}^{1,1}$ be two-dimensional Minkowski space and (N, g) be a compact Riemannian manifold. Then for any given initial data of the regularity

$$\begin{aligned}
\phi(0,x) =& \phi_0(x) \in H^2(\mathbb{R}, N), \\
\frac{\partial\phi}{\partial t}(0,x) =& \phi_1(x) \in H^1(\mathbb{R}, TN), \\
\psi(0,x) =& \psi_0(x) \in H^1(\mathbb{R}, \Sigma\mathbb{R}^{1,1} \otimes \phi^* TN) \cap W^{1,4}(\mathbb{R}, \Sigma\mathbb{R}^{1,1} \otimes \phi^* TN)
\end{aligned}$$

the Dirac-wave map equation admits a global weak solution of the class

$$\phi \in H^2(\mathbb{R}^{1,1}, N), \ \psi \in H^1(\mathbb{R}^{1,1}, \Sigma\mathbb{R}^{1,1} \otimes \phi^* TN) \cap W^{1,4}(\mathbb{R}, \Sigma\mathbb{R}^{1,1} \otimes \phi^* TN),$$

which is uniquely determined by the initial data.

- Extension to higher dimensions seems hard to achieve.

Existence of Dirac-wave maps with curvature term I

Theorem (Branding-Kröncke, 2017: Assumptions)

1. *Let* $\tilde{g}_t$ *be a smooth family of complete Riemannian metrics on* Σ^{n-1}, $N \in C^\infty(\mathbb{R} \times \Sigma)$ *with* $0 < A \leq N \leq B < \infty$ *and*

$$(M^n, \tilde{h}) = (\mathbb{R} \times \Sigma, -N^2 dt^2 + \tilde{g}_t)$$

be a globally hyperbolic manifold Lorentzian spin manifold.

2. *Assume there exists a monotonically increasing smooth function* $s : \mathbb{R} \to \mathbb{R}_+$ *with* $\int_0^\infty s^{-1} dt < \infty$*, such that the conformal metric* $h = (Ns)^{-2}\tilde{h} = -s^{-2}dt^2 + g_t$ *has bounded geometry.*
3. *Moreover, assume that*

$$\|N\|_{C^k(g_t)} + \|\nabla_\nu N\|_{C^k(g_t)} + \|\mathrm{II}\|_{C^k(g_t)} + s\|\mathrm{II}\|_{L^\infty} \leq C < \infty$$

for all $k \in \mathbb{N}$*. Here,* ν *is the future-directed unit normal of the hypersurfaces* $\{t\} \times \Sigma$ *and* II *is their second fundamental form.*

Existence of Dirac-wave maps with curvature term II

Theorem (Branding-Kröncke, 2017: Result)

Then for each $r \in \mathbb{N}$ *with* $r > \frac{n-1}{2}$ *there exists an* $\epsilon > 0$ *such that if the initial data satisfies*

$$\left\|\phi|_{t=0}\right\|_{H^{r+1}(\tilde{g}_0)} + \left\|\partial_t\phi|_{t=0}\right\|_{H^r(\tilde{g}_0)} + \left\|\psi|_{t=0}\right\|_{H^r(\tilde{g}_0)} < \epsilon,$$

the unique solution of the Dirac-wave map with curvature term system

$$\tau(\phi) = \mathcal{R}(\phi,\psi) - \frac{1}{12}\langle(\nabla R^N)^\sharp(\psi,\psi)\psi,\psi\rangle,$$
$$i\not{D}\psi = \frac{1}{3}R^N(\psi,\psi)\psi$$

exists for all times $t \in [0,\infty)$.

- First existence result for the full model!

Some comments on the result

- Gives an existence result for wave maps as well.
- In our setup it is easy to control H^r regularity of the solutions.
- Spacetimes that satisfy our assumptions:
  1. Robertson-Walker
  2. de-Sitter space
  3. power-law inflation
  4. future geodesically complete solutions of the Einstein equations with positive cosmological constant.
- Our approach does not work on Minkowski space!
- Interpretation: Our spacetime expands fast enough such that no energy concentration can happen.

Technical difficulty: Scalar products on the spinor bundle

- There exist several scalar products on ΣM when M is Lorentzian.
  1. The geometric invariant scalar product $\langle\psi,\psi\rangle$, which is not positive definite.
  2. The positive definite scalar product $\langle\psi,\partial_t\cdot\psi\rangle$, which is not invariant under the spin group. Here ∂_t is the unit timelike vector field.
  3. The physicists way: $\psi\bar{\psi}$ with $\psi\in\Gamma(\Sigma M)$ and $\bar{\psi}\in\Gamma((\Sigma M)^*)$.
- In our geometric setup

$$\nabla_{\partial_t}(s\partial_t)=0.$$

- We use the positive definite scalar product to derive estimates, but the geometric invariant scalar product in the energy functional.

Sketch of the proof

- $F_r(\phi,\psi) := s^2\|\partial_t\phi\|^2_{H^r} + \|D\phi\|^2_{H^r} + s^2\|\nabla_t\psi\|^2_{H^r} + \|D\psi\|^2_{H^r} + \|\psi\|^2_{L^2}.$
- We find the following energy inequality:

$$\begin{aligned}\frac{d}{dt}F_r(\phi,\psi) \leq & Cs^{-1}\sum_{l=0}^{2r+4}F_r(\phi,\psi)^{l/2+1} \\ & + C(n-2)\dot{s}s^{1-n}\sum_{l=0}^{r}F_r(\phi,\psi)^{l/2+2}\end{aligned}$$

- As long as $F_r(\phi,\psi)\leq 1$ we have

$$\frac{d}{dt}F_r(\phi,\psi) \leq C\big(s^{-1}(t)+(n-2)s^{1-n}\dot{s}\big)F_r(\phi,\psi).$$

- Due to the assumption on s we get a uniform bound on $F_r(\phi,\psi)$ for all times.

Existence result for uncoupled Dirac-harmonic maps

- Finally, let us come back to the case of a Riemannian domain.

Theorem (Ammann - Ginoux, 2011)

Let M be a closed Riemannian spin manifold and N a closed Riemannian manifold. Consider the homotopy class $[\phi]$ of maps $\phi\colon M \to N$ such that the ***index*** *$\alpha(M, [\phi])$ is non-trivial. Moreover, let $\phi_0 \in [\phi]$ be a harmonic map. Then there exists a linear space V such that all $(\phi_0, \psi), \psi \in V$ are Dirac-harmonic Maps.*

- The proof uses the **Atiyah-Singer index-theorem**.
- (ϕ_0, ψ) are uncoupled: $\tau(\phi_0) = 0 = \mathcal{R}(\phi_0, \psi), \qquad \not{D}\psi = 0$
- This approach still works as long as ψ solves a linear equation.

Dirac-harmonic maps from closed surfaces with boundary

- Suppose that M is a compact Riemannian surface with non-empty boundary $\partial M \neq \emptyset$ and N a compact Riemannian manifold.
- Consider the constraint heat-flow

$$\frac{\partial \phi_t}{\partial t} = \tau(\phi_t) - \mathcal{R}(\phi_t, \psi), \qquad \not{D}\psi = 0 \tag{3}$$

together with appropriate boundary-initial data.

Theorem (Jost - Liu - Zhu, 2017)

For suitable small boundary-initial data there exists a global weak solution of (3) which is smooth except finitely many singularities.
For $t \to \infty$ suitably the pair (ϕ_t, ψ) converges weakly to a Dirac-harmonic map with corresponding boundary data.

- Wittmann proved the short-time existence of (3) on closed manifolds.

Thank you for your attention!