Kähler geometry and Chern insulators

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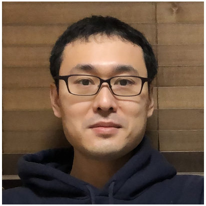
- The work presented in this talk is the result of a wonderful collaboration with Tomoki Ozawa, from the Advanced Institute for Materials Research (AIMR) in Tohoku University, Japan.



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Outline

Geometry of band insulators and Chern insulators

Quantum Hall Effect

Geometry of families of quantum states

Fractional Chern insulators and Kähler geometry

Geometry of band insulators and Chern insulators

Insulators versus metals (band theory)

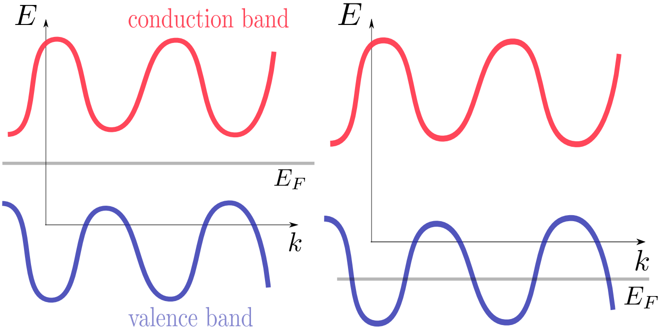


Figure: A band insulator (left) and metal (right). The ground state is obtained by filling all the states below E_F. In the insulator there is an energy gap to excite the system, while on the metal there isn't.

Setup for tight binding models

- We will consider systems of fermions on a 2-dimensional lattice with periodic boundary conditions. The period on each direction is N.
- We will eventually take the $N \to \infty$ thermodynamic limit.

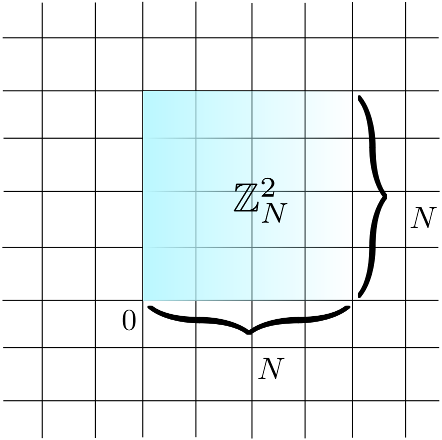


Figure: Position space.

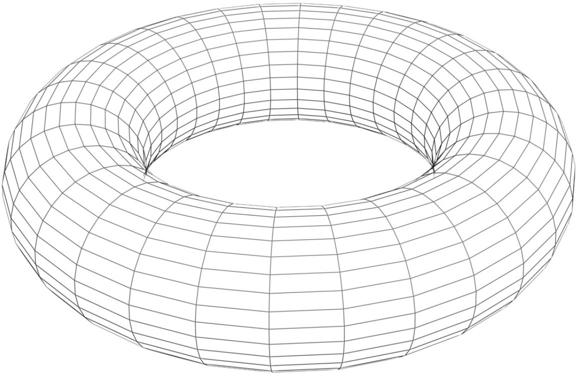


Figure: Topologically, the positions of the fermions take values in a two-torus.

- In the thermodynamic limit the real space becomes the lattice $\mathbb{Z}^2$ and the allowed momenta live in the Brillouin zone, which itself is topologically a torus $\mathrm{BZ}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$:

$$e^{i\mathbf{k}\cdot\mathbf{r}} = e^{i(\mathbf{k}+\mathbf{K})\cdot\mathbf{r}}, \text{ for } \mathbf{r} \in \mathbb{Z}^2 \text{ and } \mathbf{K} \in 2\pi\mathbb{Z}^2.$$

- With the standard trivial boundary conditions, the allowed momenta for the fermions are

$$\mathbf{k} = \frac{2\pi}{N}\mathbf{m}, \text{ with } \mathbf{m} \in \{0, ..., N-1\}^2,$$

which should be understood as taking values in BZ^2. In some appropriate sense, we recover BZ^2 as $N \to \infty$.

Band insulators and tight binding free fermion models

- In the translation invariant, charge preserving setting,

$$\mathcal{H} = \sum_{\mathbf{k}} \psi^\dagger_{\mathbf{k}} H(\mathbf{k}) \psi_{\mathbf{k}},$$

  where $\psi^\dagger_{\mathbf{k}} = [\psi^\dagger_{\mathbf{k},1} \ldots \psi^\dagger_{\mathbf{k},n}]$ is an array of fermion creation operators, accounting for internal degrees of freedom and $H(\mathbf{k})$ is an $n \times n$ Hermitian matrix yielding the action of $\mathcal{H}$ in the single particle sector.
- If the hoppings in real space decay fast enough, the map $H : \mathrm{BZ}^2 \to \mathrm{Herm}(n);\ \mathbf{k} \mapsto H(\mathbf{k})$ is smooth.

- Assume that we are in a band insulating state, so that there are bands below the Fermi level and bands above.
- The Fermi projector

$$P(\mathbf{k}) = \Theta(E_F - H(\mathbf{k}))$$

  is smooth and defines a vector bundle, the Bloch bundle $E \to \mathrm{BZ}^2$, over the Brillouin zone.
- Over each $\mathbf{k} \in \mathrm{BZ}^2$, we take the vector space of eigenvectors with energy below E_F, i.e., $E_{\mathbf{k}} = \mathrm{Im}P(\mathbf{k})$.
- Smoothness of P guarantees smoothness of E.

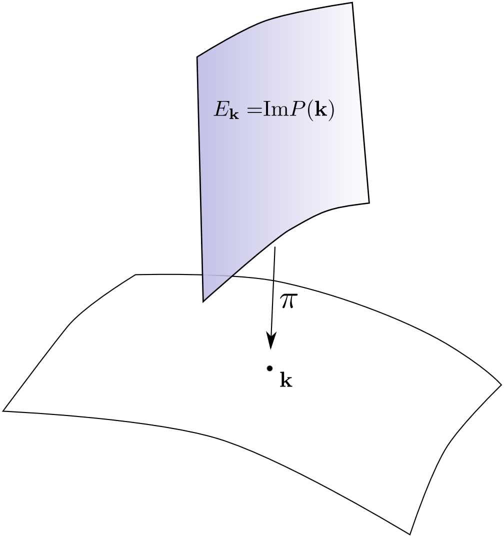


Figure: The Bloch bundle $E \to B.Z.$ defined by the valence band projector $\mathbf{k} \mapsto P(\mathbf{k})$. Notice that the $E_{\mathbf{k}}$ is naturally a subspace of a fixed vector space $\mathbb{C}^n$ since $H(\mathbf{k})$ is an $n \times n$ matrix.

Berryology

- Since each space $E_{\mathbf{k}} \subset \mathbb{C}^n$, we can define a parallel transportation rule.
- Namely, we have a connection/ covariant derivative on $E \to \mathrm{BZ}^2$, $\nabla \Psi(\mathbf{k}) = P(\mathbf{k}) d\Psi(\mathbf{k})$, for single particle wave functions Ψ on E (sections of E).

Berryology (cont.)

- Given a local o.n. basis for E provided by wave functions $\{\Psi_i\}_{i=1}^r$, the associated $\mathrm{U}(r)$ gauge field, known as the Berry gauge field, is given by

$$A = [A_{ij}] = [\langle \Psi_i | d | \Psi_j \rangle] = A_\mu dk^\mu.$$

- The Berry curvature is given by

$$F = dA + A \wedge A = [F_{ij}] = \frac{1}{2} F_{\mu\nu} dk^\mu \wedge dk^\nu,$$
$$\text{with } F_{\mu\nu} = \frac{\partial A_\nu}{\partial k^\mu} - \frac{\partial A_\mu}{\partial k^\nu} + [A_\mu, A_\nu].$$

Ground state

- Now if we are considering the finite system with periodic boundary conditions, we are sampling $H(\mathbf{k})$ at points $\mathbf{k} = (2\pi/N)\mathbf{m}$, with $\mathbf{m} \in \{0, ..., N-1\}^2$.
- The ground state is obtained by filling the bands below E_F.
- This state is constructed as follows.

Ground state (cont.)

- Forgetting about the periodicity of $H(\mathbf{k})$ in $\mathbf{k}$, we obtain a family of matrices in $\mathbb{R}^2$. Since $\mathbb{R}^2$ is contractible, we can find global assignments

$$\mathbb{R}^2 \ni \mathbf{k} \mapsto s_i(\mathbf{k}) = (a_i^1(\mathbf{k}), ..., a_i^n(\mathbf{k})) \in \mathbb{C}^n, \ i = 1, ..., r,$$

  such that for each $\mathbf{k}$, they form an o.n. of $E_{\mathbf{k}}$. The s_i's induce, generally, multivalued wave functions over the Brillouin zone.
- The s_i's give rise to creation operators (Bogoliubov-Valatin transformation)

$$\xi_{i,\mathbf{k}}^{\dagger} = \sum_{j=1}^{n} a_i^j(\mathbf{k}) \psi_{j,\mathbf{k}}^{\dagger}, \ i = 1, ..., r.$$

Ground state (cont.)

- The many-body ground state at finite size and periodic boundary conditions is

$$|GS\rangle = \prod_{\mathbf{m}\in\{0,\dots,N-1\}^2} \prod_{i=1}^{r} \xi^{\dagger}_{i,\mathbf{k}=\frac{2\pi\mathbf{m}}{N}} |0\rangle.$$

- Crucially, the piece

$$\prod_{i=1}^{r} \xi^{\dagger}_{i,\mathbf{k}=\frac{2\pi\mathbf{m}}{N}} |0\rangle$$

can be identified with a generating element of the top exterior power of the fiber $E_{\mathbf{k}}$, namely

$$s_1(\mathbf{k}) \wedge \cdots \wedge s_r(\mathbf{k}) \in \Lambda^r E_{\mathbf{k}}.$$

- The ground state and its physical observable properties are completely captured by the map

$$f : \mathrm{BZ}^2 \to \mathbb{P}(\Lambda^r \mathbb{C}^n)\,;\; \mathbf{k} \mapsto [s_1(\mathbf{k}) \wedge \cdots \wedge s_r(\mathbf{k})].$$

$$\begin{tikzcd}
\mathrm{BZ}^2 \arrow[r, "f"] \arrow[dr, "P"'] & \mathbb{P}(\Lambda^r \mathbb{C}^n) \\
 & \mathrm{Gr}_r(\mathbb{C}^n) \arrow[u, "\iota"']
\end{tikzcd}$$

- f is the composition of two maps $f = \iota \circ P$.
- $P : \mathrm{BZ}^2 \to \mathrm{Gr}_r(\mathbb{C}^n);\; \mathbf{k} \mapsto P(\mathbf{k})$, where we identify the set of orthogonal projectors of rank r in $\mathbb{C}^n$ with the Grassmannian of r-planes in $\mathbb{C}^n$.
- The second map ι is the *Plücker embedding* which sends an r-dimensional vector subspace $E \subset \mathbb{C}^n$ to the line through the wedge product of a basis E.

Response to an external gauge field – Chern insulators

- Typically, one is interested in the response of an insulator to an external electromagnetic field.
- Linear response to a uniform electric field assumes the form

$$J^i = \sigma^i_j E^j,$$

where $\sigma = [\sigma^i_j]$ is the DC conductivity tensor.

- The DC conductivity tensor may have a purely anti-symmetric contribution which gives rise to the quantum Hall effect:

$$J^i = \varepsilon_{ij}\sigma_{\mathrm{Hall}} E^j .$$

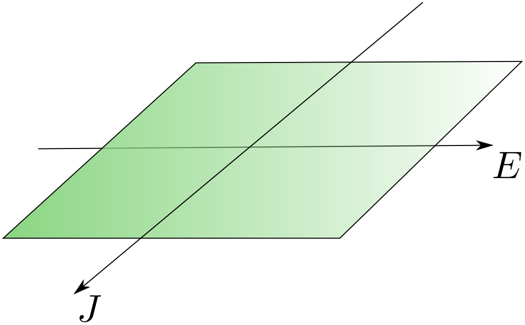


Figure: The anomalous Hall effect.

- A beautiful result by Thouless-Kohmoto-Nightingale-den Nijs is that the Hall conductivity of the insulator is a topological invariant:

$$\sigma_{\mathrm{Hall}} = \frac{e^2}{h} \underbrace{c_1(E) \cdot [\mathrm{BZ}^2]}_{\text{characteristic number of } E} = \frac{e^2}{h} \int_{\mathrm{BZ}^2} \mathrm{Tr}\left(\frac{iF}{2\pi}\right)$$

- Unlike in the Quantum Hall effect, we do not need an external magnetic field to have non-vanishing σ_{Hall} – anomalous Hall effect / Chern insulator.
- This phenomenon is stable up to adiabatic deformations preserving the gap. Mathematically: it depends only on the homotopy class of $P : \mathrm{BZ}^2 \to \mathrm{Gr}_r(\mathbb{C}^n)$.
[actually, of $f : \mathrm{BZ}^2 \to \mathbb{P}(\Lambda^r \mathbb{C}^n)$]

- Chern insulators are then $2d$ band insulators described by maps $P : \mathrm{BZ}^2 \to \mathrm{Gr}_r(\mathbb{C}^n)$ with non-trivial topological charge yielding $\sigma_{\mathrm{Hall}} \neq 0$.
- Chern insulating phases are topological phases of *free fermions*.
- Arguably more interesting topological phases are those of *strongly interacting fermions*.
- A paradigmatic example is the FQHE – ground state degeneracy tied to the topology of the base manifold; anyonic gapped excitations above the ground state with fractional charges.
- To realize such phases in band insulators it is useful to understand what are the ingredients that allow for the FQHE in the first place.

Quantum Hall Effect

LLL physics

- When one considers the single particle theory of electrons in the plane in the presence of an external uniform magnetic field one gets the so-called Landau levels — they correspond to energy levels which are infinitely degenerate.
- The lowest Landau level (LLL) can be interpreted as the space of square integrable holomorphic sections of the electromagnetic line bundle where the electromagnetic gauge field acts as a connection.

$$\psi_m(z) \sim e^{-\frac{B|z|^2}{4}} z^m, \; m \in \mathbb{N}.$$

LLL and geometric quantization

- In geometric quantization terms, the LLL is the quantization of $(\mathbb{R}^2, Bdx \wedge dy)$ equipped with the Kähler structure coming from the magnetic field 2$-$form $iF = Bdx \wedge dy$ and the standard complex structure determined by $z = x + iy$.
- Effective reduction of the classical phase space $T^*\mathbb{R}^2 \cong \mathbb{R}^4$ to the "magnetic" plane $\mathbb{R}^2$.

- Filling the LLL, i.e., filling fraction $\nu = 1$, one obtains a many-body state which accurately describes the integer Hall effect

$$\Psi(z_1, ..., z_N) = \underbrace{\prod_{i<j}(z_i - z_j)e^{-\sum_i \frac{|z_i|^2}{4}}}_{\text{Slater determinant}}$$

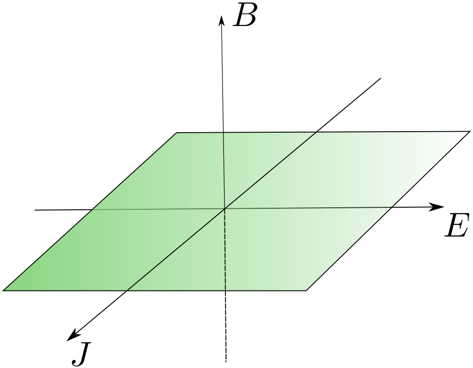


Figure: The integer quantum Hall effect in the plane. Linear response described by action $S(A) = (1/4\pi)\nu \int AdA$. [$A$ the external gauge field.]

- For large B, for $\nu = 1/m$, the Coulomb interaction is expected to mix the states within the Landau levels, but not expected to mix Landau levels $\implies$ can project to the LLL $P_{LLL} : \mathcal{H} \to \text{LLL}$.
- Effective Hamiltonian described in terms of the projected density operators

$$P_{LLL}\rho_{\mathbf{q}}P_{LLL} = \overline{\rho}_{\mathbf{q}}, \text{ with } \rho_{\mathbf{q}} = \sum_{\mathbf{k}} \psi^{\dagger}_{\mathbf{k}+\mathbf{q}}\psi_{\mathbf{k}}.$$

  [In first quantized language $\rho_{\mathbf{q}} = e^{i\mathbf{q}\cdot\mathbf{r}}$.]
- $$H = \frac{1}{2}\sum_{\mathbf{q}} V(\mathbf{q})\overline{\rho}_{-\mathbf{q}}\overline{\rho}_{\mathbf{q}},$$

  where the $V(\mathbf{q})$'s are the Fourier components of the Coulomb potential. No kinetic part because we are projecting to the LLL.

- The projected density operators satisfy the GMP/ $W_\infty-$algebra

$$[\overline{\rho}_{\mathbf{q}_1}, \overline{\rho}_{\mathbf{q}_2}] = 2ie^{\frac{B}{2}\mathbf{q}_1\cdot\mathbf{q}_2} \sin\left(\frac{B}{2}\mathbf{q}_1 \times \mathbf{q}_2\right) \overline{\rho}_{\mathbf{q}_1+\mathbf{q}_2}$$

which is a quantum version of the algebra of area-preserving diffeomorphisms of the plane equipped with (magnetic) symplectic form $Bdx \wedge dy$: $\{e^{i\mathbf{q}_1\cdot\mathbf{r}}, e^{i\mathbf{q}_2\cdot\mathbf{r}}\} = B\mathbf{q}_1 \times \mathbf{q}_2 e^{i(\mathbf{q}_1+\mathbf{q}_2)\cdot\mathbf{r}}$
- This is consistent with the fact that the LLL is the quantization of the symplectic plane.

- Unfortunately, there is no known exact GS solution to the strong interacting problem described above.
- Magically, for $\nu = 1/m$, we have the Laughlin trial ground state wave function

$$\Psi_\nu(z_1, ..., z_N) = \prod_{i<j}(z_i - z_j)^m e^{-\sum_i \frac{|z_i|^2}{4}},$$

  which very well describes experiments. Such ground state wave function is argued to be in the same phase as the ground state wave function of the FQHE Hamiltonian (adiabatically connected).
- Furthermore, the quasihole excitations described by

$$\Psi_\xi(z_1, ..., z_N) = \prod_{i=1}^{N}(\xi - z_i)\Psi_\nu(z_1, ..., z_N)$$

  have charges which are $1/m$ of the electron charge. They also have fractional exchange statistics – anyons.

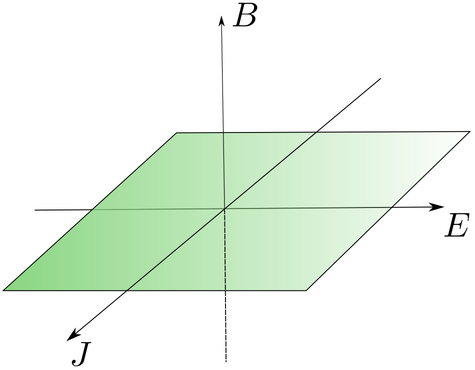


Figure: The fractional quantum Hall effect in the plane. Linear response described by action $S(A, a) = -(1/4\pi)\nu^{-1} \int ada + (1/2\pi) \int Ada$. [$a$ is the *internal statistical gauge field* and A the external gauge field.]

- To summarize, the FQHE appears in the context of a fractionally filled single particle flat band and the effective projected Hamiltonian

$$H = \frac{1}{2}\sum_{\mathbf{q}} V(\mathbf{q})\overline{\rho}_{-\mathbf{q}}\overline{\rho}_{\mathbf{q}},$$

where the $\overline{\rho}_{\mathbf{q}}$ satisfy the algebra

$$[\rho_{\mathbf{q}_1}, \rho_{\mathbf{q}_2}] = 2ie^{\mathbf{q}_1\cdot\mathbf{q}_2}\sin\left(\frac{B}{2}\mathbf{q}_1 \times \mathbf{q}_2\right)\rho_{\mathbf{q}_1+\mathbf{q}_2}.$$

Geometry of families of quantum states

Families of quantum states

- A state in quantum mechanics is one-dimensional subspace of a Hilbert space $\mathcal{H}$.
- Hence, the set of quantum states is $\mathbb{P}(\mathcal{H})$.
- Assume, for now, $\mathcal{H} = \mathbb{C}^N$ and hence $\mathbb{P}(\mathcal{H}) = \mathbb{C}P^{N-1}$.
- A family of quantum states parameterized by some manifold M is a map $f : M \to \mathbb{C}P^{N-1}$.

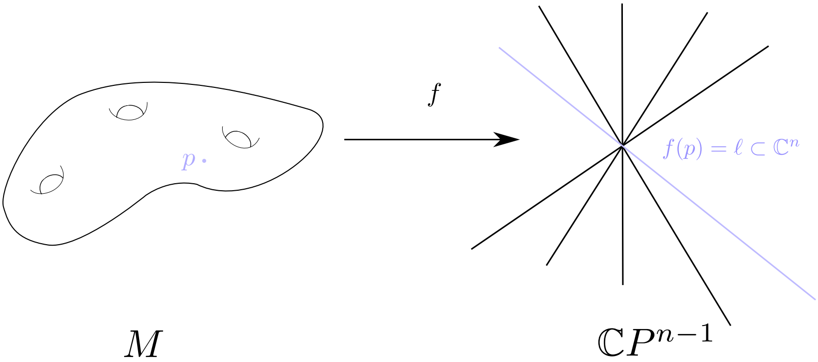


Figure: A family of quantum states.

Families of quantum states (cont.)

- Associated with f we have a line bundle over M whose fiber at $p \in M$ is $f(p) \subset \mathbb{C}^N$.
- Over $\mathbb{C}P^{N-1}$ we have a "tautological" family of quantum states $f = \mathrm{id}_{\mathbb{C}P^{N-1}}$.
- The associated line bundle is the tautological line bundle $L \to \mathbb{C}P^{N-1}$, whose fiber over $\ell \subset \mathbb{C}^N$ is ℓ itself.
- We then see that the line bundle associated to $f : M \to \mathbb{C}P^{N-1}$ is the pullback f^*L.

Families of quantum states (cont.)

- L is a subbundle of the trivial bundle $\mathbb{C}P^{N-1} \times \mathbb{C}^N$.
- Can define a connection ∇ by orthogonal projection. This connection, and its pullback $f^*\nabla$ acting on f^*L, are known in physics as *Berry connections.*
- If $s : U \subset \mathbb{C}P^{N-1} \to L$ is a local section of L, we can see it as an assignment $\ell \mapsto |\psi(\ell)\rangle \in \ell \subset \mathbb{C}^N$, and then, locally,

$$\mathrm{curv}(\nabla) = F = \frac{\langle d\psi| \left(1 - \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}\right) |d\psi\rangle}{\langle\psi|\psi\rangle}.$$

Kähler structure

- The projective space $\mathbb{C}P^{N-1}$ is naturally a Kähler manifold w.r.t. the Fubini-Study triple of structures $(\omega_{FS}, J_{FS}, g_{FS})$ where ω_{FS} is a symplectic form, J_{FS} is an integrable complex structure and g_{FS} is a Riemannian metric.
- The three structures form a compatible triple:

$$\omega_{FS}(\cdot, J_{FS}\cdot) = g_{FS}.$$

- The structure of Kähler manifold is closely related to $L \to \mathbb{C}P^{N-1}$:

$$\omega_{FS} = -\frac{i}{2}F.$$

Normalization: $\int_{\mathbb{C}P^1} \omega_{FS} = -\pi c_1(L) \cdot [\mathbb{C}P^1] = \pi$.

- In quantum mechanics, typically, the family $f : M \to \mathbb{C}P^{N-1}$ arises from the eigenvalue problem of a smooth family of Hamiltonians $H : M \to \mathrm{Herm}(N)$,

$$f(p) = \ker\left(H(p) - \varepsilon_n(p) I_N\right) = \mathrm{span}_{\mathbb{C}}\left\{|n(p)\rangle\right\},$$

then

$$\mathrm{curv}(f^*\nabla) = f^*F = \sum_{m \neq n} \frac{\langle n|dH|m\rangle \wedge \langle m|dH|n\rangle}{(\varepsilon_m - \varepsilon_n)^2},$$

which connects to familiar perturbation theory formulae.

Geometry of orthogonal projectors

- It is enlightening to cast the previous statements in terms of orthogonal projectors in $\mathbb{C}^N$.
- A quantum state $|\psi\rangle \sim \lambda|\psi\rangle$, $\lambda \in \mathbb{C}^*$, is uniquely specified by an orthogonal projector

$$P = \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}.$$

- We can then identify $\mathbb{C}P^{N-1}$ with the set of orthogonal projectors of rank 1.
- A family of quantum states is then a family of orthogonal projectors of rank 1: $\{P(p)\}_{p\in M}$.

Geometry of orthogonal projectors (cont.)

- Since orthogonal projectors satisfy $P^2 = P$, we have, by differentiation,

$$PdPP = QdPQ = 0,$$

where $Q = I - P$ is the orthogonal complement projector.

- Furthermore, since $P^\dagger = P$, we have that dP is completely determined by $QdPP$, i.e., a lower triangular matrix in the basis where $P = \text{diag}(1, 0, 0, ..., 0)$.

Geometry of orthogonal projectors (cont.)

- The line bundle L has fiber $\ell = \mathrm{Im} P$. A section of L satisfies $Ps(\ell) = s(\ell)$ (where inclusion in the trivial bundle is implicit). The Berry connection is explicitly

$$\nabla s = Pds.$$

- Then, the curvature is the endomorphism of L determined by

$$\begin{aligned} \nabla \wedge \nabla(s) &= Pd(Pds) = PdP \wedge ds = PdP \wedge d(Ps) \\ &= (PdP \wedge dPP)\, s. \end{aligned}$$

  To get the concrete 2-form, we may trace, to obtain

$$\begin{aligned} \mathrm{curv}(\nabla) &= \mathrm{Tr}\,(PdP \wedge dPP) \\ &= \mathrm{Tr}\,(PdPQ \wedge QdPP) \\ &= \mathrm{Tr}\left[(QdPP)^\dagger \wedge QdPP\right]. \end{aligned}$$

The Fubini-Study Kähler structure in terms of orthogonal projectors

- We learn that

$$\omega_{FS} = -\frac{i}{2}\mathrm{Tr}\left[(QdPP)^\dagger \wedge QdPP\right] = -\frac{i}{2}\mathrm{Tr}\left(PdP \wedge dP\right).$$

- It can be shown that

$$(QdPP) \circ J_{FS} = iQdPP.$$

[Think of the LHS as a matrix valued one-form in $\mathbb{C}P^{n-1}$]

- Finally,

$$g_{FS} = \omega_{FS}(\cdot, J_{FS}\cdot) = \mathrm{Tr}\left(PdPdP\right).$$

Quantum geometric tensor

- Can organize the data in terms of an Hermitian metric on $\mathbb{C}P^{N-1}$

$$\chi = g_{FS} + i\omega_{FS} = \mathrm{Tr}\left(PdP \otimes dP\right).$$

- The quantities $f^*\chi$, $f^*F = 2if^*\omega_{FS}$ and f^*g_{FS} are known in the physics community as the *quantum geometric tensor*, the *Berry curvature* and the *quantum metric*, respectively.
- Note that although g_{FS} is a Riemannian metric, f^*g_{FS} is not necessarily so, as df may not be full rank everywhere.
- These geometric quantities can actually be measured in the Lab! [See N. Goldman, T. Ozawa, *et al.* [1], [2]]

Measuring the quantum geometric tensor

- To do so let $H : M \to \mathrm{Herm}(N)$ be a family of Hamiltonians, where M is the parameter manifold for which a certain energy level $\varepsilon_n(p)$ describes our family of quantum states $|n(p)\rangle$, $p \in M$.
- Fix a point $p \in M$ (may be couplings, external fields,...) described by local coords x^μ, $\mu = 1, ..., m = \dim M$ and prepare the system in the eigenstate $|n(p)\rangle$.

Measuring the quantum geometric tensor (cont.)

- We may then periodically modulate the system as

$$x^{\mu}(t) = x^{\mu} + (2E/\Omega)\cos(\Omega t)\delta^{\mu\nu},$$

  for $E << \Omega$.
- This gives rise to a time dependent Hamiltonian $H(x(t))$,

$$H(x(t)) \approx H(x) + \frac{2E}{\Omega}\cos(\Omega t)\frac{\partial H}{\partial x^{\nu}}(x).$$

  and we may apply time dependent perturbation theory to get the corresponding Fermi Golden rule

$$\Gamma(\Omega) \approx 2\pi E^2 \sum_{m\neq n} \frac{|\langle n|\frac{\partial H}{\partial x^{\nu}}|m\rangle|^2}{\Omega^2}\delta(E_n - E_m - \Omega)$$

Measuring the quantum geometric tensor (cont.)

- Integration over Ω yields the quantum metric

$$\int d\Omega\, \Gamma(\Omega) = 2\pi E^2 g_{\nu\nu}(x).$$

- Different protocols of periodic modulation can be used to extract the other components of the quantum metric and even the Berry curvature – thus, a full tomography of the quantum geometric tensor is possible.

Fractional Chern insulators and Kähler geometry

- To look for FQHE on Chern insulators, we begin with a dispersionless Chern band described by $P : \mathrm{BZ}^2 \to \mathbb{C}P^{n-1}$.
- Turn on density-density interactions:

$$V = \sum_{i<j} V(\mathbf{r}_i - \mathbf{r}_j),$$

where the $\mathbf{r}$'s are the positions of the fermions on the lattice.

- The scale of the interaction is assumed to be smaller than the gap separating the Chern band from other bands $\implies$ can project the interaction to the Chern band.

- The position operators on $\mathbb{Z}^2$ act, in momentum space as $x^i = \sqrt{-1}\frac{\partial}{\partial k_i}$.
- For small transferred momentum $\mathbf{q}$, the projected density operator, in first quantized notation, acts as

$$\begin{aligned} \overline{\rho}_{\mathbf{q}} &= P(\mathbf{k})e^{i\mathbf{q}\cdot\mathbf{r}}P(\mathbf{k}) = P(\mathbf{k})e^{\mathbf{q}\cdot\frac{\partial}{\partial \mathbf{k}}}P(\mathbf{k}) \\ &\approx 1 + q_i\nabla_i - \frac{1}{2}q_iq_jPx^ix^jP \\ &= 1 + \nabla_q + \frac{1}{2}\nabla_q\nabla_q - \frac{1}{2}|q|^2 \end{aligned}$$

where $\nabla = PdP$ is the Berry connection, $q = q_i\frac{\partial}{\partial k_i}$ and $|q|^2 = g(q,q)$, where g is the quantum metric.
[The operator $\overline{\rho}_{\mathbf{q}}$ acts on sections of the Bloch bundle $L \to \mathrm{BZ}^2$].

- A tedious but straightforward calculation shows that, up to third order,

$$
\begin{aligned}
[\overline{\rho}_{\mathbf{q}_1}, \overline{\rho}_{\mathbf{q}_2}] \approx{} & F(q_1, q_2)(1 + \nabla_{q_1+q_2}) \\
& - \frac{1}{2}\left(\mathcal{L}_{q_1+q_2} F(q_1, q_2) + \mathcal{L}_{q_1}|q_2|^2 - \mathcal{L}_{q_2}|q_1|^2\right),
\end{aligned}
$$

  where $\mathcal{L}_q$ denotes the Lie derivative with respect to q.
- If the quantum geometric tensor is flat, we then get an algebra of projected density operators

$$
[\overline{\rho}_{\mathbf{q}_1}, \overline{\rho}_{\mathbf{q}_2}] \approx F(q_1, q_2)\overline{\rho}_{\mathbf{q}_1+\mathbf{q}_2}.
$$

- If, furthermore, the Berry curvature and the quantum metric satisfy

$$\frac{|F_{12}|}{2} = \sqrt{\det(g)}$$

it can be shown [Roy [3]] that the projected density operators satisfy exactly

$$[\overline{\rho}_{\mathbf{q}_1}, \overline{\rho}_{\mathbf{q}_2}] = 2ie^{g(q_1,q_2)} \sin\left(\frac{1}{2}|F(q_1, q_2)|\right) \overline{\rho}_{\mathbf{q}_1+\mathbf{q}_2}.$$

- The condition

$$\frac{|F_{12}|}{2} = \sqrt{\det(g)},$$

  for non-vanishing $\sqrt{\det(g)}$ is mathematically equivalent to the map $P : \mathrm{BZ}^2 \to \mathbb{C}P^{n-1}$ being holomorphic! But holomorphic with respect to what complex structure?
- If $\sqrt{\det(g)} \neq 0$, then P must be an immersion (because $P^* g_{FS}$ is a Riemannian metric). Then, by taking an orientation on BZ^2, we may, by 90 degree rotation define a complex structure J. Since BZ^2 is 2-dimensional, J is integrable and turns BZ^2 into a complex manifold.

- Then P being holomorphic means
$$dP \circ J = J_{FS} \circ dP$$
- This then implies that the triple $(\omega, J, g) = (P^*\omega_{FS}, J, P^*g)$ is compatible
$$\omega(\cdot, J\cdot) = g.$$
- Since $\omega = -iF/2$, by taking determinants, we have
$$\omega_{12} = \frac{|F_{12}|}{2} = \sqrt{\det(g)}.$$
- We have proved the statement in one direction, the other direction can be proved using the Wirtinger inequality and choosing the appropriate orientation.
[Since $\omega = -iF/2$ and $\mathcal{C} = \int_{\mathrm{BZ}^2} iF/2\pi$ it is the one that makes the 1st Chern number negative].

Flat Kähler bands: ideal for FCIs

- We then look for Chern bands for which the defining map $P : \mathrm{BZ}^2 \to \mathbb{C}P^{n-1}$ is a *holomorphic immersion*, hence (ω, J, g) is a compatible triple, and such that the Kähler structure is *flat*.
- Such bands are natural candidates for hosting fractional Chern insulating phases!

Engineering flat Kähler bands

- There is a natural way to engineer flat Kähler bands in an appropriate limit.
- The way to do this, is using the mathematical framework of geometric quantization and Bergman kernels.
- The idea is that if we fix a flat Kähler structure (ω, J, g) on BZ^2 and pick an Hermitian holomorphic line bundle $L \to \mathrm{BZ}^2$ such that the Chern connection satisfies

$$\mathrm{curv}(\nabla) = -i\omega,$$

  then by picking an L^2-orthonormal basis of $H^0(\mathrm{BZ}^2, L^{\otimes p})$, for large enough p, described by sections $\{s_1, ..., s_p\}$ [Riemann-Roch] the map

$$f : \mathrm{BZ}^2 \to \mathbb{C}P^{p-1};\ \mathbf{k} \mapsto [a_1(\mathbf{k}) : \cdots : a_p(\mathbf{k})]$$

  is an holomorphic immersion and

$$f^*\omega_{FS} - p\omega \to 0,\ \text{as } p \to \infty.$$

Bergman kernel asymptotics

- More precisely,

$$f^*\omega_{FS} - p\omega = -\frac{i}{2}\partial\overline{\partial}\log B,$$

where

$$B = \sum_{j=1}^{p} h^p|a_j|^2 = \sum_{j=1}^{p} h^p(s_j, s_j),$$

is the diagonal of the Bergman kernel, that has the asymptotic expansion

$$B = p + A_1 p^0 + A_2 p^{-1} + \cdots + A_k p^{1-k} + \dots,$$

where A_k's are smooth functions involving geometric invariants.

Explicit construction

- We then fix a flat complex structure on the torus described by $z = (k_1 + \tau k_2)/2\pi$, $\tau \in \mathcal{H}$.
- We begin by choosing a basic Hermitian holomorphic line bundle $L \to \mathrm{BZ}^2$ such that $\mathrm{curv}(\nabla) = (1/2\pi) dk_1 \wedge dk_2$.
- A natural choice is the one for which the unique holomorphic section is described by the Jacobi theta function

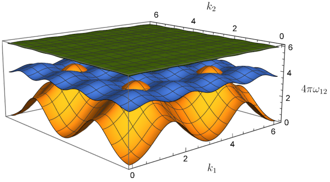
$$\theta(z, \tau) = \sum_{n \in \mathbb{Z}} e^{i\pi\tau n^2 + 2\pi i n z}$$

- The holomorphic sections of $L^{\otimes p}$ are described by theta functions with characteristics

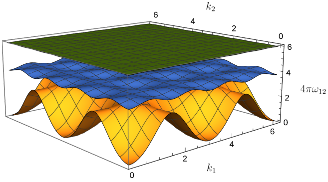
$$a_j(z) = \vartheta \left[\begin{array}{c} \frac{j}{p} \\ 0 \end{array} \right] (pz, p\tau)$$
$$= \sum_{n \in \mathbb{Z}} e^{i\frac{\pi}{p}\tau(j+np)^2 + 2\pi i(j+np)z}, \; j = 0, ..., p-1.$$

- We then have a sequence of Chern bands $f_p : \mathrm{BZ}^2 \to \mathbb{C}P^{p-1}$ which are asymptotically flat Kähler bands.

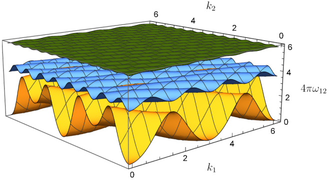
Asymptotic flatness results



(a) $\tau = i$



(b) $\tau = e^{i\pi/3}$



(c) $\tau = 2e^{i\pi/7}$

Figure: $f_p^*\omega_{FS}$ as a function of the quasimomentum $\mathbf{k} \in \mathrm{BZ}^2$, for $p = 2$ (orange), $p = 4$ (blue), $p = 6$ (green), for various anisotropies τ. $\sqrt{\det(g)}dk_1 \wedge dk_2$ overlaps exactly for all the cases shown here.

- Physically, one can then construct tight-binding models by writing

$$H(\mathbf{k}) = I - 2P(\mathbf{k}), \text{ with } \langle i|P(\mathbf{k})|j\rangle = \frac{\overline{a_i(z)}a_j(z)}{\sum_k |a_k(z)|^2}$$

[note that this corresponds to two flat bands with energies ± 1] and then Fourier transforming to real space

$$H(\mathbf{r}_i, \mathbf{r}_j) = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{r}_i - \mathbf{r}_j)} H(\mathbf{k}).$$

- However, such models are not very physical and easy to realize in the Lab because they involve infinite-range hopping.
- Therefore, it is natural to ask what happens when we truncate the hoppings to nearest neighbours, next-to-nearest neighbours, and so on.

Results on truncation of hoppings

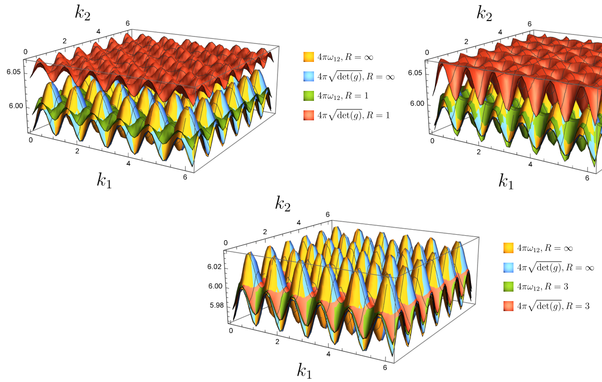


Figure: The geometrical structure, $\sqrt{\det(g)}$ and ω_{12}, of truncated models as a function of $\mathbf{k} \in \mathrm{BZ}^2$, for $p = 6$. In the top panel $R = 1$ is compared to the long range case $R = \infty$; in the middle panel $R = 2$ is compared to the $R = \infty$ case, and in the bottom panel $R = 3$ is compared to $R = \infty$.

Discussion

- $\sqrt{\det(g)}$ and ω_{12} are no longer equal $\implies$ breaking of holomorphicity of the Chern bands. However, the difference between $\sqrt{\det(g)}$ and ω_{12} is not so large even for the case of nearest-neighbor model.
- The flatness of the geometrical quantities changes. The geometrical quantities will not become more dispersive, and sometimes they can become even *flatter* when truncating the hopping.
- We note that, even when $R = 1$ and $R = 2$, the Chern number, is the same as $R = \infty \implies$ the bands are adiabatically connected to the ideal Kähler band.
- We have also numerically checked the cases with smaller p and the overall behavior remains the same.

- I want to remark that in [4], we have also shown the no-go theorem that it is impossible to construct exactly flat Kähler bands with a finite total number of bands, i.e., from a holomorphic map $f : \mathrm{BZ}^2 \to \mathbb{C}P^{p-1}$ for finite p.

If you want to read more and go through details check our papers: [5, 6, 4].

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Thank you!

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